

Application of fuzzy Laplace transforms for solving fuzzy partial Volterra integro-differential equations

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Abstract

Fuzzy partial integro-differential equations have a major role in the fields of science and engineering. In this paper, we propose the solution of fuzzy partial Volterra integro-differential equation with convolution type kernel using fuzzy Laplace transform method (FLT) under Hukuhara differentiability. It is shown that FLT is a simple and reliable approach for solving such equations analytically. Finally, the method is illustrated with few examples to show the ability of the proposed method.

Keywords: Fuzzy valued function, fuzzy partial differential equation, fuzzy Laplace transform, fuzzy convolution, fuzzy partial Volterra integro-differential equation.

1 Introduction

The topic of fuzzy integro differential equations (FIDEs) has been rapidly grown recent years. The basic idea and arithmetics of fuzzy sets were first introduced by L. A. Zadeh in [31]. The concept of fuzzy derivatives and fuzzy integration were studied in [11, 27] and then some generalization have been investigated in [10, 11, 24, 23]. One of the most important field of the fuzzy theory is the fuzzy differential equations [1, 14], fuzzy integral equations [2, 6, 12] and fuzzy integro-differential equations (FIDEs) [15, 16, 20]. The FIDEs is obtained when a physical system is modeled under differential sense [18]. Also FIDEs in fuzzy setting are a natural way to model uncertainty of dynamical systems. Therefore the solution of the fuzzy integro-differential equations is very important in various fields such as Physics, Geographic, Medical and Biological Sciences [5, 32, 22]. In [27] Seikkala defined fuzzy derivatives while concept of integration of fuzzy functions was first introduced by Dubois and Prade [11]. Alternative

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approaches were later studied in [17, 19]. The idea of fuzzy partial differential equations (FPDEs) was first introduced by Buckley in [8]. Allahveranloo proposed the difference method for solving FPDEs in [3].

In [4] Allahveranloo and Salahshour proposed the idea of fuzzy Laplace transform method for solving first order fuzzy differential equations under generalized H-differentiability. The technique of fuzzy Laplace transform method to solve fuzzy convolution Volterra integral equations (FCVIEs) of the second kind was developed in [26]. Recently the technique used in [26] was extended for solving fuzzy convolution Volterra integro differential equations (FCVIDEs) in [29] under generalized Hukuhara differentiability.

In [28] the solution of classical PIDEs was discussed using classical Laplace transform. In the present article we investigate the solution of different types of fuzzy partial integro differential equations with convolution kernel (FPIDEs) using fuzzy Laplace transform method. In order to determine the lower and upper functions of the solution we convert the given FPIDEs to two crisp ordinary differential equations by using FLT.

Rest of the paper is organized as follows:

In section 2, some basic definitions and results are stated which will be used throughout this paper. In section 3, two dimensional fuzzy Laplace transform is given and fuzzy convolution theorem is stated in this case. In section 4, the fuzzy Laplace transform is applied to fuzzy partial Volterra integro-differential equation to construct the general technique. Illustrative examples are also considered to show the ability of the proposed method in section 5, and the conclusion is drawn in section 6.

2 Preliminaries

In this section we will recall some basic definitions and theorems needed throughout the paper such as fuzzy number, fuzzy-valued function and the derivative of the fuzzy-valued functions [13, 31].

Definition 2.1. A fuzzy number is defined as the mapping such that $u : R \rightarrow [0, 1]$, which satisfies the following four properties

1. u is upper semi-continuous.
2. u is fuzzy convex that is $u(\lambda x + (1 - \lambda)y) \geq \min \{u(x), u(y)\}$. $x, y \in R$ and $\lambda \in [0, 1]$.
3. u is normal that is $\exists x_0 \in R$, where $u(x_0) = 1$.
4. $A = \{\overline{x \in \mathbb{R} : u(x) > 0}\}$ is compact, where \overline{A} is closure of A .

Definition 2.2. A fuzzy number in parametric form is given as an order pair of the form $u = (\underline{u}(r), \overline{u}(r))$, where $0 \leq r \leq 1$ satisfying the following conditions.

1. $\underline{u}(r)$ is a bounded left continuous increasing function in the interval $[0, 1]$.
2. $\overline{u}(r)$ is a bounded left continuous decreasing function in the interval $[0, 1]$.
3. $\underline{u}(r) \leq \overline{u}(r)$.

If $\underline{u}(r) = \overline{u}(r) = r$, then r is called crisp number.

Since each $y \in R$ can be regarded as a fuzzy number if

$$\tilde{y}(t) = \begin{cases} 1, & \text{if } y = t, \\ 0, & \text{if } y \neq t. \end{cases}$$

For arbitrary fuzzy numbers $u = (\underline{u}(\alpha), \overline{u}(\alpha))$ and $v = (\underline{v}(\alpha), \overline{v}(\alpha))$ and an arbitrary crisp number j , we define addition and scalar multiplication as:

1. $(\underline{u} + \underline{v})(\alpha) = (\underline{u}(\alpha) + \underline{v}(\alpha)).$
2. $(\overline{u} + \overline{v})(\alpha) = (\overline{u}(\alpha) + \overline{v}(\alpha)).$
3. $(j\underline{u})(\alpha) = j\underline{u}(\alpha), (j\overline{u})(\alpha) = j\overline{u}(\alpha), \quad j \geq 0.$
4. $(j\underline{u})(\alpha) = j\overline{u}(\alpha)\alpha, (j\overline{u})(\alpha) = j\underline{u}(\alpha)\alpha, \quad j < 0.$

Definition 2.3. (See [4, 7]) Let us suppose that $x, y \in E$, if $\exists z \in E$ such that $x = y + z$, then z is called the H -difference of x and y and is given by $x \ominus y$.

The Housdorff distance between the fuzzy numbers [15, 24, 4, 7] defined by

$$d : E \times E \longrightarrow R^+ \cup \{0\},$$

$$d(u, v) = \sup_{r \in [0, 1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\},$$

where $u = (\underline{u}(r), \overline{u}(r))$ and $v = (\underline{v}(r), \overline{v}(r)) \subset R$.

We know that if d is a metric in E , then it will satisfy the following properties, introduced by Puri and Ralescu [21]:

1. $d(u + w, v + w) = d(u, v), \forall u, v, w \in E.$
2. $(k \odot u, k \odot v) = |k|d(u, v), \forall k \in R, \text{ and } u, v \in E.$
3. $d(u \oplus v, w \oplus e) \leq d(u, w) + d(v, e), \forall u, v, w, e \in E.$

Theorem 2.4. (see Wu [30]) Let f be a fuzzy-valued function on $[a, \infty)$ given in the parametric form as $(\underline{f}(x, r), \overline{f}(x, r))$ for any constant number $r \in [0, 1]$. Here we assume that $\underline{f}(x, r)$ and $\overline{f}(x, r)$ are Riemann-Integrable on $[a, b]$ for every $b \geq a$. Also we assume that $\underline{M}(r)$ and $\overline{M}(r)$ are two positive functions, such that $\int_a^b |\underline{f}(x, r)|dx \leq \underline{M}(r)$ and $\int_a^b |\overline{f}(x, r)|dx \leq \overline{M}(r)$ for every $b \geq a$, then $f(x)$ is improper fuzzy Riemann-integrable on $[a, \infty)$. Thus an improper integral will always be a fuzzy number. In short

$$\int_a^b f(x)dx = \left(\int_a^b \underline{f}(x, r)dx, \int_a^b \overline{f}(x, r)dx \right).$$

Next we define the n th order partial H -derivatives for fuzzy valued functions $u = u(x, t)$ with respect to x and t in a similar way as given in [25, 9].

Definition 2.5. The function $u : (a, b) \times (a, b) \rightarrow E$, is said to be H -differentiable of the n th order at $t_0 \in (a, b)$, with respect to t , if \exists an element $\frac{\partial^n}{\partial t^n} u(x, t_0) \in E$ such that

1. $\forall h > 0$ sufficiently small $\exists \frac{\partial^{n-1}}{\partial t^{n-1}}u(x, t_0+h) \ominus \frac{\partial^{n-1}}{\partial t^{n-1}}u(x, t_0), \frac{\partial^{n-1}}{\partial t^{n-1}}u(x, t_0) \ominus \frac{\partial^{n-1}}{\partial t^{n-1}}u(x, t_0-h)$, then the following limits hold (in the metric d)

$$\lim_{h \rightarrow 0} \frac{\frac{\partial^{n-1}}{\partial t^{n-1}}u(x, t_0+h) \ominus \frac{\partial^{n-1}}{\partial t^{n-1}}u(x, t_0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{\partial^{n-1}}{\partial t^{n-1}}u(x, t_0) \ominus \frac{\partial^{n-1}}{\partial t^{n-1}}u(x, t_0-h)}{h} = \frac{\partial^n}{\partial t^n}u(x, t_0), \text{ or}$$
2. $\forall h > 0$ sufficiently small $\exists \frac{\partial^{n-1}}{\partial t^{n-1}}u(x, t_0) \ominus \frac{\partial^{n-1}}{\partial t^{n-1}}u(x, t_0+h), \frac{\partial^{n-1}}{\partial t^{n-1}}u(x, t_0-h) \ominus \frac{\partial^{n-1}}{\partial t^{n-1}}u(x, t_0)$, then the following limits hold (in the metric d)

$$\lim_{h \rightarrow 0} \frac{\frac{\partial^{n-1}}{\partial t^{n-1}}u(x, t_0) \ominus \frac{\partial^{n-1}}{\partial t^{n-1}}u(x, t_0+h)}{-h} = \lim_{h \rightarrow 0} \frac{\frac{\partial^{n-1}}{\partial t^{n-1}}u(x, t_0-h) \ominus \frac{\partial^{n-1}}{\partial t^{n-1}}u(x, t_0)}{-h} = \frac{\partial^n}{\partial t^n}u(x, t_0).$$

Similarly

Definition 2.6. The function $u : (a, b) \times (a, b) \rightarrow E$, is said to be H -differentiable of the n th order at $x_0 \in (a, b)$, w.r.t x , if \exists an element $\frac{\partial^n}{\partial x^n}u(x_0, t) \in E$ such that

1. $\forall h > 0$ sufficiently small $\exists \frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0+h, t) \ominus \frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0, t), \frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0, t) \ominus \frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0-h, t)$, then the following limits hold (in the metric d)

$$\lim_{h \rightarrow 0} \frac{\frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0+h, t) \ominus \frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0, t)}{h} = \lim_{h \rightarrow 0} \frac{\frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0, t) \ominus \frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0-h, t)}{h} = \frac{\partial^n}{\partial x^n}u(x_0, t), \text{ or}$$
2. $\forall h > 0$ sufficiently small $\exists \frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0, t) \ominus \frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0+h, t), \frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0-h, t) \ominus \frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0, t)$, then the following limits hold (in the metric d)

$$\lim_{h \rightarrow 0} \frac{\frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0, t) \ominus \frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0+h, t)}{-h} = \lim_{h \rightarrow 0} \frac{\frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0-h, t) \ominus \frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0, t)}{-h} = \frac{\partial^n}{\partial x^n}u(x_0, t).$$

The denominators h and $-h$ denote multiplication by $\frac{1}{h}$ and $\frac{-1}{h}$ respectively.

3 Two dimensional fuzzy Laplace transform

In this section we state some definitions and theorems from [25, 26] which will be used in the next section.

Definition 3.1. Let $u = u(x, t)$ is a fuzzy-valued function and p is a real parameter, then FLT of the function u with respect to t denoted by $U(x, p)$, is defined as follows:

$$U(x, p) = L[u(x, t)] = \int_0^\infty e^{-pt} u(x, t) dt = \lim_{\tau \rightarrow \infty} \int_0^\tau e^{-pt} u(x, t) dt,$$

$$U(x, p) = \left[\lim_{\tau \rightarrow \infty} \int_0^\tau e^{-pt} \underline{u}(x, t) dt, \lim_{\tau \rightarrow \infty} \int_0^\tau e^{-pt} \overline{u}(x, t) dt \right],$$

whenever the limits exist. The r -cut representation of $U(x, p)$ is given as:

$$U(x, p; r) = L[u(x, t; r)] = [l(\underline{u}(x, t; r)), l(\overline{u}(x, t; r))],$$

where

$$l[\underline{u}(x, t; r)] = \int_0^\infty e^{-pt} \underline{u}(x, t; r) dt = \lim_{\tau \rightarrow \infty} \int_0^\tau e^{-pt} \underline{u}(x, t; r) dt,$$

$$l[\overline{u}(x, t; r)] = \int_0^\infty e^{-pt} \overline{u}(x, t; r) dt = \lim_{\tau \rightarrow \infty} \int_0^\tau e^{-pt} \overline{u}(x, t; r) dt.$$

Now to use FLT method we have to state the following result:

Theorem 3.2. *Let $u : (a, b) \times (a, b) \longrightarrow E$ is a fuzzy valued function such that its derivatives up to $(n - 1)$ th order w.r.t “ t ” are continuous for all $t > 0$ and u^n exists then*

$$L\left[\frac{\partial^n}{\partial x^n} u(x, t)\right] = p^n U(x, p) \ominus p^{n-1} u(x, 0) \ominus p^{n-2} u(x, 0) \ominus \dots \ominus u^{n-1}(x, 0) \quad (3.1)$$

$$L\left[\frac{\partial^n u}{\partial x^n}\right] = \frac{d^n}{dx^n} L[u(x, t)] = \frac{d^n}{dx^n} U(x, p), \quad (3.2)$$

Definition 3.3. *The fuzzy two dimensional convolution of fuzzy-valued functions f and g defined by*

$$(f * g)(t) = \int_0^t f(s)g(t-s)ds,$$

where $t > 0$ and it exists if f and g are say, piecewise continues functions.

Theorem 3.4. *(Fuzzy convolution theorem), if f and g are piecewise continuous fuzzy-valued function on $[0, \infty)$, and of exponential order p , then*

$$L[(f * g)(t)] = L[f(t)].L[g(t)].$$

4 Constructing the propose method

In this section, we will investigate solution of fuzzy convolution partial Volterra integro-differential equation using fuzzy Laplace transform. Consider the most general equation FPIDE

$$\sum_{i=0}^m a_i \frac{\partial^i u}{\partial x^i} + \sum_{i=0}^n b_i \frac{\partial^i u}{\partial t^i} + u + f(x, t) = \int_0^t k(t-s)u(x, s)ds \quad (4.1)$$

With some appropriate fuzzy intial and boundary conditions. Also the functions $f(x, t)$ is given fuzzy valued function while $k(t, s)$ is given crisp kernel and $a'_i s$, $b'_i s$ are constants or functions of x .

Applying Fuzzy Laplace on both side of (4.1) with respect to t we get:

$$\sum_{i=0}^m a_i L\left[\frac{\partial^i u}{\partial x^i}\right] + \sum_{i=0}^n b_i L\left[\frac{\partial^i u}{\partial t^i}\right] + cL[u] + L[f(x, t)] = L\left[\int_0^t k(t-s)u(x, s)ds\right] \quad (4.2)$$

Using fuzzy convolution theorem 3.4 on integral part we get

$$\sum_{i=0}^m a_i L\left[\frac{\partial^i u}{\partial x^i}\right] + \sum_{i=0}^n b_i L\left[\frac{\partial^i u}{\partial t^i}\right] + cL[u] + L[f(x, t)] = L[k(t)]L[u(x, t)] \quad (4.3)$$

Now using the definition of fuzzy Laplace stated in section 3, equation (4.3) becomes

$$\sum_{i=0}^m a_i \frac{d^i U(x, p)}{dx^i} + \sum_{i=0}^n b_i U(x, p) \ominus \sum_{j=1}^i p^{j-1} b_i u^{i-j}(x, 0) + U(x, p) + L[f(x, t)] = L[k(t)]U(x, p). \quad (4.4)$$

The classical form of (4.4) gives the following two m^{th} order ordinary differential equations as follows:

$$\sum_{i=0}^m a_i \frac{d^i \underline{U}(x, p, r)}{dx^i} + \sum_{i=0}^n b_i \underline{U}(x, p, r) - \sum_{j=1}^i p^{j-1} b_i \underline{u}^{(i-j)}(x, 0, r) + \underline{U}(x, p, r) + \underline{F}(x, t, r) = K(t) \underline{U}(x, p, r). \quad (4.5)$$

$$\sum_{i=0}^m a_i \frac{d^i \overline{U}(x, p, r)}{dx^i} + \sum_{i=0}^n b_i \overline{U}(x, p, r) - \sum_{j=1}^i p^{j-1} b_i \overline{u}^{(i-j)}(x, 0, r) + \overline{U}(x, p, r) + \overline{F}(x, t, r) = K(t) \overline{U}(x, p, r). \quad (4.6)$$

Where $K(t) = L[k(t)]$ and $F(x, t) = L[f(x, t)]$. Using the given fuzzy initial and boundary conditions the upper and lower solutions of (4.1) can be find out from equations (4.5) and (4.6) respectively.

5 Numerical examples

In this section we will discuss the solution of fuzzy convolution partial Volterra integro-differential equations using FLT to show the utility of the proposed method in Section 4.

Example 5.1. *Let us consider the following fuzzy convolution partial Volterra integro-differential equation.*

$$xu_x = u_{tt} + (x \sin x)(r-1, 1-r) + \int_0^t \sin(t-s)u(x, s)ds, \quad (5.1)$$

with initial conditions

$$u(x, 0, r) = (0, 0), \quad u_t(x, 0, r) = ((r-1)x, (1-r)x),$$

and Boundary condition

$$u(1, t, r) = ((r-1)t, (1-r)t).$$

Taking FLT with respect to t on (5.1), then we get

$$xL[u_x] = L[u_{tt}] + x(r-1, 1-r)L[\sin t] + L[\sin t]L[u(x, t)]. \quad (5.2)$$

Using FLT (5.2) becomes

$$x \frac{d}{dx} U(x, p) = p^2 U(x, p) \ominus p u(x, 0) \ominus u_t(x, 0) + \frac{x}{p^2 + 1} (r-1, 1-r) + \frac{1}{p^2 + 1} U(x, p). \quad (5.3)$$

Also applying FLT boundary condition becomes

$$U(1, p; r) = \frac{(r-1, 1-r)}{p^2}. \quad (5.4)$$

The r -cut representation of (5.3) after using initials conditions is given by

$$x \frac{d}{dx} \underline{U}(x, p, r) = p^2 \underline{U}(x, p, r) - (r-1)x + \frac{x}{p^2+1}(r-1) + \frac{1}{p^2+1} \underline{U}(x, p, r), \quad (5.5)$$

and

$$x \frac{d}{dx} \overline{U}(x, p, r) = p^2 \overline{U}(x, p, r) - (1-r)x + \frac{x}{p^2+1}(1-r) + \frac{1}{p^2+1} \overline{U}(x, p, r). \quad (5.6)$$

From (5.5) we get:

$$\frac{d}{dx} \underline{U}(x, p, r) - \frac{p^4 + p^2 + 1}{x(p^2 + 1)} \underline{U}(x, p, r) + \frac{(r-1)p^2}{p^2 + 1} = 0, \quad (5.7)$$

Solving (5.7) we get

$$\underline{U}(x, p, r) = \frac{(r-1)}{p^2} x + C x^{\frac{p^4 + p^2 + 1}{p^2 + 1}}. \quad (5.8)$$

On using boundary condition given in (5.4) we get, $C = 0$ therefore (5.8) become

$$\underline{U}(x, p, r) = \frac{(r-1)}{p^2} x \quad (5.9)$$

Finally taking inverse Laplace on both side of (5.9)

$$\underline{u}(x, t, r) = (r-1)xt$$

Similarly on simplifying (5.6) the following differential equation is obtained:

$$\frac{d}{dx} \overline{U}(x, p, r) - \frac{p^4 + p^2 + 1}{x(p^2 + 1)} \overline{U}(x, p, r) + \frac{(1-r)p^2}{p^2 + 1} = 0, \quad (5.10)$$

Which gives the final upper solution of (5.1) as follows:

$$\overline{u}(x, t, r) = (1-r)xt$$

Example 5.2. Let us consider the following FPVIDE

$$u_x = u_{tt} + 2(1+r, 3-r)e^x - 2 \int_0^t (t-s)u(x, s)ds, \quad (5.11)$$

with initial conditions

$$u_t(x, 0, r) = (0, 0), \quad u(x, 0, r) = e^x((1+r), (3-r)),$$

and Boundary condition

$$u(0, t, r) = \cos t((r+1), (3-r)).$$

Applying FLT on (5.11), we have

$$L[u_x] = L[u_{tt}] + 2(r-1, 1-r)e^x L[1] - 2L[t]L[u(x, t)]. \quad (5.12)$$

Using definition of FLT (5.12) becomes

$$\frac{d}{dx} U(x, p) = p^2 U(x, p) \ominus p u(x, 0) \ominus u_t(x, 0) + \frac{2e^x}{p}(r+1, 3-r) - \frac{2}{p^2} U(x, p). \quad (5.13)$$

After using FLT boundary condition gives:

$$U(0, p, r) = \frac{(r+1, 3-r)p}{p^2+1}. \quad (5.14)$$

The classical form of (5.13) after using initial conditions, is

$$\frac{d}{dx} \underline{U}(x, p, r) = p^2 \underline{U}(x, p, r) - pe^x(r+1) + \frac{2e^x}{p^2+1}(r+1) - \frac{2\underline{U}(x, p, r)}{p^2}, \quad (5.15)$$

and

$$\frac{d}{dx} \overline{U}(x, p, r) = p^2 \overline{U}(x, p, r) - pe^x(3-r) + \frac{2e^x}{p^2+1}(3-r) - \frac{2\overline{U}(x, p, r)}{p^2}, \quad (5.16)$$

Now solving (5.15) and (5.16) after using boundary condition (5.14) we get:

$$\underline{U}(x, p, r) = (r+1)e^x \left(\frac{p^3 - 2p}{p^4 - p^2 - 2} \right) \quad (5.17)$$

$$\overline{U}(x, p, r) = (3-r)e^x \left(\frac{p^3 - 2p}{p^4 - p^2 - 2} \right) \quad (5.18)$$

Finally taking inverse Laplace we get the lower and upper solutions as follow:

$$\underline{u}(x, t, r) = (r+1)e^x \cos t$$

$$\overline{u}(x, t, r) = (3-r)e^x \cos t$$

Example 5.3. Let us consider the following FPVIDE

$$u_{xx} = u_t + u + (1+r, 3-r)\{-(x^2+1)e^t + 2\} + \int_0^t e^{(t-s)}u(x, s)ds, \quad (5.19)$$

with given conditions

$$u(x, 0, r) = (r+1, 3-r)x^2, \quad u_t(x, 0, r) = (1+r, 3-r),$$

and

$$u(0, t, r) = (r+1, 3-r)t, \quad u_x(0, t, r) = (0, 0).$$

Applying FLT on (5.19), we have

$$L[u_{xx}] = L[u_t] + L[u] + (r+1, 3-r)\{-(x^2+1)L[e^t] + 2L[1]\} + L[e^t]L[u(x, t)]. \quad (5.20)$$

Using definition of FLT (5.20) becomes

$$\frac{d^2}{dx^2} U(x, p) = pU(x, p) \ominus u(x, 0) + U(x, p) + \left[\frac{-(x^2+1)}{p-1} + \frac{2}{p} \right] (r+1, 3-r) + \frac{U(x, p)}{p-1}, \quad (5.21)$$

while the transform boundary conditions are:

$$U(0, p, r) = \frac{(r+1, 3-r)}{p^2}, \quad U_x(0, p, r) = (0, 0)$$

The classical form of (5.21) after using initial conditions, is as under

$$\frac{d^2}{dx^2} \underline{U}(x, p, r) = \left(\frac{p^2}{p-1} \right) \underline{U}(x, p, r) - (1+r)[x^2 + \frac{(x^2+1)}{p-1}] + \frac{2(1+r)}{p}, \quad (5.22)$$

and

$$\frac{d^2}{dx^2}\overline{U}(x, p, r) = \left(\frac{p^2}{p-1}\right)\overline{U}(x, p, r) - (3-r)\left[x^2 + \frac{(x^2+1)}{p-1}\right] + \frac{2(3-r)}{p}, \quad (5.23)$$

Solving (5.22) we get:

$$\underline{U}(x, p, r) = c_1 e^{\sqrt{\frac{p^2}{p-1}}x} + c_2 e^{-\sqrt{\frac{p^2}{p-1}}x} + \frac{(1+r)x^2}{p} + \frac{(1+r)}{p^2}, \quad (5.24)$$

Using the transform fuzzy boundary conditions we have $c_1 = 0$ and $c_2 = 0$. Hence (5.24) becomes:

$$\underline{U}(x, p, r) = \frac{(1+r)x^2}{p} + \frac{(1+r)}{p^2}. \quad (5.25)$$

Finally taking inverse Laplace we get

$$\underline{u}(x, t, r) = (1+r)[x^2 + t].$$

Similarly solving (5.23) in same way we have:

$$\overline{u}(x, t, r) = (3-r)[x^2 + t].$$

6 Conclusion

In this paper we investigated the applicability of fuzzy Laplace transform for the solution of FPVIDEs under H-differentiability with crisp kernel. In our knowledge this is the first attempt toward the solution of such equations with fuzzy conditions. We have illustrated the method by solving some examples. In future we will discuss the solution of FPVIDEs under generalized H-differentiability with both crisp and fuzzy kernel.

References

- [1] S. Abbasbandy, T. Allahviranloo, Oscar Lopez-Pouso, and J. J. Nieto. Numerical method for solving fuzzy differential inclusion. *Computers & Mathematics with Applications*, 48(10-11):1633–1641, 2004.
- [2] S. Abbasbandy, E. Babolian, and M. Alavi. Numerical method for solving linear fredholm fuzzy integral equations of the second kind. *Chaos, Solitons & Fractals*, 31(1):138–146, 2007.
- [3] T. Allahveranloo. Difference methods for fuzzy partial differential equations. *Computational methods in applied mathematics*, 2(3):233–242, 2002.
- [4] T. Allahviranloo and M. Barkhordari Ahmadi. Fuzzy laplace transforms. *Soft Computing*, 14(3):235–243, 2010.
- [5] H. F. Arnoldus. Application of the magnetic field integral equation to diffraction and reection by a conducting sheet. *International Journal of Theoretical Physics, Group Theory and Nonlinear Optics*, 14(3):1–12, 2011.

- [6] E. Babolian, H. Sadeghi Goghary, and S. Abbasbandy. Numerical solution of linear fredholm fuzzy integral equations of the second kind by adomian method. *Applied Mathematics & Computation*, 161(3):733–744, 2005.
- [7] B. Bede and S. G. Gal. Generalizations of the differentiability of fuzzy-number- valued functions with applications to fuzzy differential equations. *Fuzzy Sets & Systems*, 151(3):581–599, 2005.
- [8] J. J. Buckley and T. Feuring. Introduction to fuzzy partial differential equations. *Fuzzy Sets and Systems*, 105:241–248, 1999.
- [9] Y. Chalco-Cano and H. Román-Flores. On new solutions of fuzzy differential equations. *Chaos, Solitons & Fractals*, 38(1):112–119, 2008.
- [10] Y. Chalco-Cano and H. Roman-Flores. On new solutions of fuzzy differential equations. *Chaos, Solitons & Fractals*, 38(1):112–119, 2008.
- [11] D. Dubois and H. Prada. Towards fuzzy differential calculus part 1: Integration of fuzzy mappings. *Journal of Approximation Theory*, 8(1):1–17, 1982.
- [12] M. Friedman, M. Ma, and A. Kandel. Numerical solutions of fuzzy differential and integral equations. *Fuzzy Sets & Systems*, 106(1):35–48, 1999.
- [13] R. Goetschel Jr and W. Voxman. Elementary fuzzy calculus. *Fuzzy Sets & Systems*, 18(1):31–43, 1986.
- [14] O. Kaleva. Fuzzy differential equations. *Fuzzy Sets & Systems*, 24(3):301–317, 1987.
- [15] Y. C. Kwun, M. J. Kim, B. Y. Lee, and J. H. Park. Existence of solutions for the semilinear fuzzy integro-differential equations using by successive iteration. *Journal of Korean Institute of Intelligent Systems*, 18:543–548, 2008.
- [16] Y. C. Kwun, M. J. Kim, J. S. Park, and J. H. Park. Continuously initial observability for the semilinear fuzzy integro-differential equations. *Proceedings of the 5th International Conference on Fuzzy Systems and Knowledge Discovery (FSKD) Jinan, China*, 1:225–229, 2008.
- [17] M. Matloka. On fuzzy integrals. *Proc. 2nd Polish Symp. on Interval and Fuzzy Mathematics, Politechnika Poznansk*, pages 167–170, 1987.
- [18] J. Mordeson and W. Newman. Fuzzy integral equations. *Information Sciences*, 87(4):215–229, 1995.
- [19] S. Nanda. On integration of fuzzy mappings. *Fuzzy Sets & Systems*, 32(1):95–101, 1989.
- [20] J. H. Park, J. S. Park, and Y. C. Kwun. Controllability for the semilinear fuzzy integrodifferential equations with nonlocal conditions. In Lipo Wang, Licheng Jiao, Guanming Shi, Xue Li, and Jing Liu, editors, *Fuzzy Systems and Knowledge Discovery*, volume 4223 of *Lecture Notes in Computer Science*, pages 221–230. Springer Berlin, Heidelberg, 2006.

- [21] M. L. Puri and D. A. Ralescu. Fuzzy random variables. *Journal of Mathematical Analysis & Applications*, 114:409–422, 1986.
- [22] Sheng quan Ma, Fu chuan Chem, and Zhi qing Zhao. Choquet type fuzzy complex-values integral and its application in classification. In Xian-Jung Xie Bing-Yuan Cao, editor, *Fuzzy Engineering & Operation Research*, volume 147, pages 229–237. Springer-Verlag, Berlin Heidelberg, Germany, 2012.
- [23] R. Rodríguez-López. Comparison results for fuzzy differential equations. *Information Sciences*, 178(6):1756–1779, 2008.
- [24] L.J. Rodríguez-Muñiz and M. López-Díaz. Hukuhara derivative of the fuzzy expected value. *Fuzzy Sets & Systems*, 138(3):593–600, 2003.
- [25] S. Salahshour and E. Haghi. Solving fuzzy heat equation by fuzzy laplace transforms. *Information Processing and Management of Uncertainty in Knowledge-Based Systems. Applications Communications in Computer and Information*, 81:512–521, 2010.
- [26] S. Salahshour, M. Khezerloo, S. Hajighasemi, and M. Khorasany. Solving fuzzy integral equations of the second kind by fuzzy laplace transform method. *Int. J. Industrial Mathematics*, 4(1):21–29, 2012.
- [27] S. Seikkala. On the fuzzy initial value problem. *Fuzzy Sets & Systems*, 24(3):319–330, 1987.
- [28] J. Thorwe and S. Bhalekar. Solving partial integro-differential equations using laplace transform method. *American Journal of Computational and Applied Mathematics*, 2(3):101–104, 2012.
- [29] S. Ullah, L. Ahmad, M. Farooq, and S. Abdullah. Solving fuzzy voltterra integro differential equations via fuzzy laplace transforms. *Submitted*.
- [30] H. C. Wu. *The improper fuzzy Riemann integral and its numerical integration*, volume 111. 1998.
- [31] L. A. Zadeh. Fuzzy sets. *Information & Control*, 8(3):338–353, 1965.
- [32] S. Zhang, Y. Zhang, and I. Gutman. Analysis of DNA sequences based on the fuzzy integral. *Communications in Mathematical and in Computer Chemistry*, 70:413–430, 2013.